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A COMPARISON OF THE FALLOUT MASS-SIZE DISTRIBUTIONS CALCULATED BY LOGNORMAL AND POWER-LAW MODELS

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ABSTRACT

Fallout mass-size distributions presently used at USNRDL are compared with new distributions suggested by recent investigations. Available data is unable to define the distribution parameters well enough to distinguish between lognormal and power-law distribution models.

SUMMARY

Problem

To determine the differences between newly-proposed fallout mass-size distributions and distributions now in use.

Findings

Differences are trivial in comparison with the effects of current uncertainties in the distribution parameters. Planned sensitivity analyses need not explicitly treat the newly-proposed power-law distributions except as a matter of convenience.

INTRODUCTION

The purpose of this report is to compare formulas for the size distribution of particle mass in the fallout from land-surface bursts. The distribution equations to be compared are basically the newly-proposed power-law distribution and the lognormal distribution, although modifications of each are involved. Cumulative forms of the distributions will be emphasized. Interest in such a comparison arises from the need to assess the impact of new information and suggestions (mainly due to Russell)[1] on present prediction techniques. The mass-size distribution is of basic importance for predicting fractionation effects because, being equivalent to the volume-size distribution, it is primary input data.

At this Laboratory both total particle mass and active particle mass are considered to have the same lognormal distribution, even though there may be some fifteen times as much inactive material as active material.[2] The power-law distribution, proposed by Russell,[1] refers only to total mass. The land-surfaces involved are both silicate (Nevada Test Site-NTS) and coral (Eniwetok Proving Grounds-EPG).

The results of the mathematical comparison offered here will throw light upon the need of a sensitivity analysis for more detailed comparison.

BACKGROUND

For purposes of fallout prediction, the information required for input can be conveniently discussed in terms of cumulative logarithmic-probability graphs because any single-valued distribution curve on such a graph is automatically normalized to 100 %. Let us say that the predictor is interested in the distribution of Mo^{99} in a local fallout field. Even though his computer can reasonably handle hundreds of particle-size classes, the data on which his input is based need not be so detailed. Presumably he knows enough about the device to state the total quantity of Mo^{99} present. His next most urgent need is to know what fraction of this came down locally, or at early times, or in large particles. This corresponds to placing a point on the graph in the region of 25 to 50-μ diameter particles. Such a point might conceivably come from several sources, e.g.,

1. Prediction techniques in vogue,
2. Partition inferred from the radiochemical analysis of cloud and ground samples,
3. Integration of contours of Mo^{99} surface density on the ground.

Assuming a number is available to place this point, he still needs 199 more numbers. These can be obtained from

1. Prediction techniques in vogue,
2. Determinining Mo^{99} distribution with particle size in a representative sample of the local field,
3. Integration of contours of particle surface density as a function of size, plus a knowledge of the relative Mo^{99} content of different particle-size classes,
4. Determining Mo^{99} distribution with particle size in a cloud sample which contains representative proportions both of the particles chosen above and of larger particles.

The more observational data that can be incorporated, the less reliance need be placed on choice one and moreover, the better the quality of choice one that can be offered. If as few as five well-distributed points were available from observations, the predictor would be in good shape.

These same considerations apply to the distribution of other particle properties: individual radionuclides, mass, or fraction of the unpartitioned, unfractionated normalization factor. Unfortunately, it is invariably necessary to rely on choice one to some extent. It is therefore incumbent upon the predictor to keep current the comparison of his techniques with published observations.

THE LOGNORMAL DISTRIBUTION

To illustrate the notation we may introduce the lognormal distribution by saying that, if the probability is $P_M\ (a,b)$ that a randomly chosen particle in a distribution has a mass x: $a < x \leq b$, then the equations

$$\left.\begin{aligned} P_M \{a,b\} &= \int_a^b p_M\ (x)\ dx \\ P_M \{x,x+dx\} &= p_M\ (x)\ dx \end{aligned}\right\}$$

define the probability density function $p_M(x)$, and for the lognormal distribution,

$$p_M(x)\ dx = \frac{1}{\sqrt{2\pi}}\ \exp \left\{ - \frac{1}{2} \left(\frac{\ln x - \mu_M}{\sigma} \right)^2 \right\} \frac{d \ln x}{\sigma}\ . \qquad (1)$$

The parameters μ_M and σ are best determined from a cumulative log-probability plot of the mass distribution. If a straight line gives a reasonable fit to the cumulative weight, then

$$\mu_M = \ln \underline{x}_M$$

(where $\underline{x}_M$ is the particle diameter at 50 %)

$$\sigma = \ln \frac{x_{M+}}{\underline{x}_M} = \ln \frac{\underline{x}_M}{x_{M-}}$$

where x_{M+} and x_{M-} are the particle diameters one sigma unit larger and smaller than $\underline{x}_M$ (i.e., at 84.13 % and 15.87 %, respectively). The mass distribution currently in use at NRDL uses the lognormal distribution with parameters $\underline{x}_M = 100\ \mu$ and $\sigma = 1.68?$.

The mass-distribution curve favored at NRDL is based upon (1) the assumption of constant volume-specific activity and (2) activity-size data from 75 to 3300 μ.[2] An argument can be made for assuming constant surface-specific activity instead of constant volume-specific activity. The effect of this will be discussed below.

THE POWER DISTRIBUTION

In an unclassified section of a classified report,[1] Russell describes his analysis of Johny Boy cloud samples, which analyses led him to the conclusion that the mass of the debris was so distributed that, down to about 90 μ, equal size increments contained equal masses:

$$P_M \{a,b\} \propto (b-a)$$

or

$$p_M = \text{constant}$$

On the assumption of constant particle density and the usual spherical-particle approximation, this leads to a frequency distribution function

$$P_N \{x,x+dx\} = p_N (x)dx = k_N x^{-3} dx$$

where k_N is a normalization factor.

Further investigation, especially by Nathans,* indicated variations from the power of three, so for generality we write

$$P_N \{x,x+dx\} = k_N x^{-q} dx$$

$$P_M \{x,x+dx\} = k_N p_M(x)\ dx = k_M x^{3-q}\ dx$$

which bears little resemblance to Equation 1. The value of q appears to lie between 3 and 4.

*M. W. Nathans, Tracerlab, Inc., private communication.

Completion of the distribution equation requires that some upper limit x_{max} be set to the particle size. Thus the form of the equation becomes

$$P_M \{a,b\} = \int_a^b x^{3-q}\, dx \Bigg/ \int_0^{x_{max}} x^{3-q}\, dx$$

Various considerations led Russell to choose a value of 1000 μ for x_{max}, although References 2 and 3 indicate a higher value would be more appropriate.

Figure 1 compares the power distribution with the lognormal distribution for several cases. The power distribution is plotted for typical values of $q = 3.0$ and 3.5, both with a cutoff of $x_{max} = 1000\ \mu$. A curve for $q = 3.0$ and $x_{max} = 2000\ \mu$ shows the effect of a reasonable change in cutoff diameter.

For the case of $q = 3.0$ and $x_{max} = 1000\ \mu$ the mass distribution is extremely simple. The fraction of mass in particles of diameter x or less is then

$$P_M \{0,x\} = \frac{x(\mu)}{1000}$$

For the case of $q = 3.5$ and $x_{max} = 1000\ \mu$ it is not much more complicated:

$$P_M \{0,x\} = \int_0^x x^{-1/2}\, dx \Bigg/ \int_0^{1000} x^{-1/2}\, dx$$

$$= \sqrt{\frac{x(\mu)}{1000}}$$

For the lognormal distributions the equations are derived according to the method of Reference 4:

$$P_M \{0,x\} = \frac{1}{1.682\sqrt{2\pi}} \int_0^x \exp\left\{-\frac{1}{2}\left(\frac{\ln\, [x(\text{microns})/100]}{1.682}\right)^2\right\} d \ln x$$

for constant volume-specific activity and

$$P_M \{0,x\} = \frac{1}{1.682\sqrt{2\pi}} \int_0^x \exp\left\{-\frac{1}{2}\left(\frac{\ln\, [x(\text{microns})/1693]}{1.682}\right)^2\right\} d \ln x$$

for constant surface-specific activity.

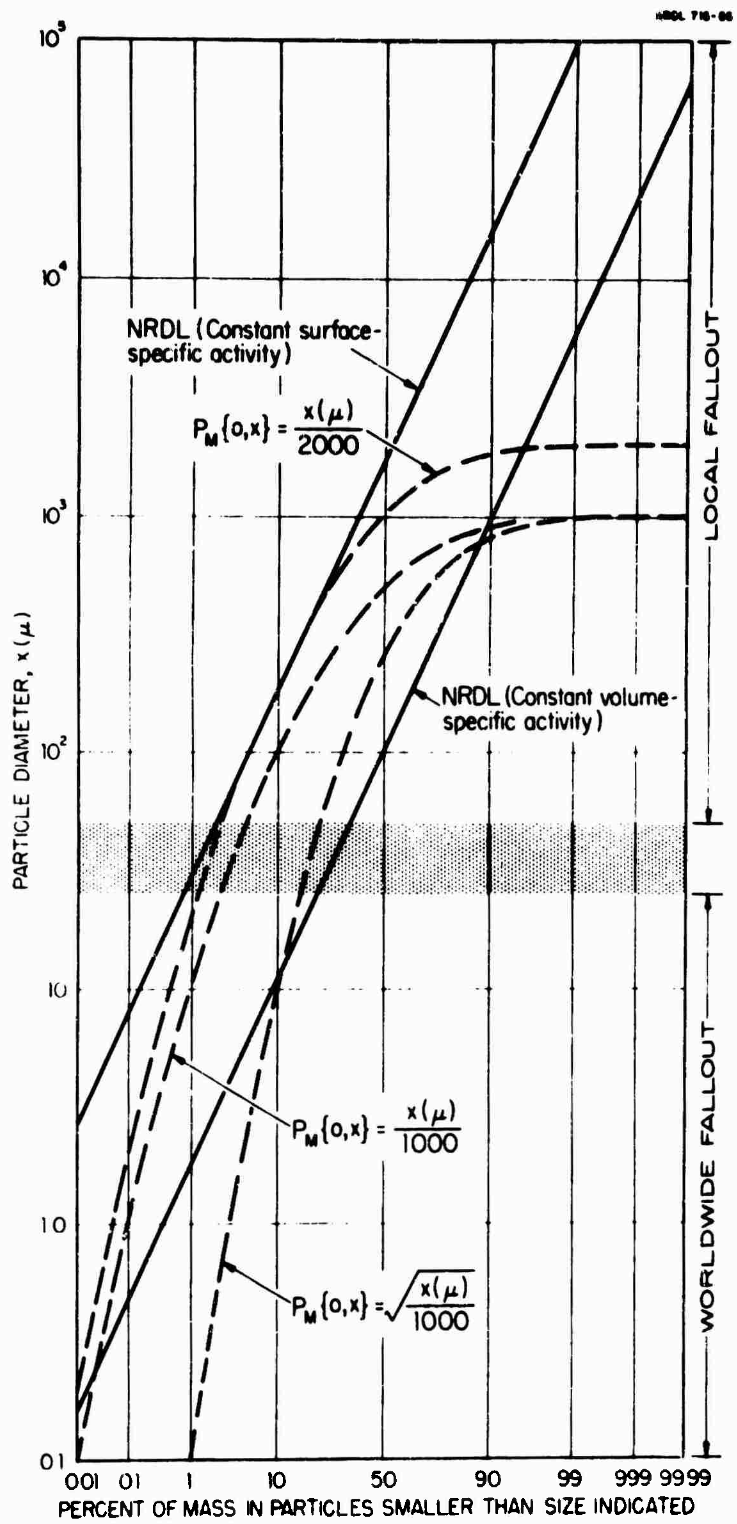

Fig. 1 Comparison of Extended Distributions

THE TRUNCATED DISTRIBUTIONS

The Truncated Power Law

It was immediately evident to Russell that the power-law distribution was physically unrealistic as shown. Thus as $x \to 0$, the distribution predicts an infinite number of particles. Therefore Russell chose a value of 1 μ for the lower cutoff diameter x_{min}, giving the final form of the distribution as

$$P_M \{a,b\} = \frac{\int_a^b x^{3-q}\, dx}{\int_1^{1000} x^{3-q}\, dx}$$

The small size cutoff has since been well documented by Nathans' work.

The Truncated Lognormal Distribution

If pressed to the limit, the lognormal distribution will predict some finite (if fractional) number of particles of any size, no matter how large or small. A means of truncating the lognormal distribution at the small particle sizes x_{min} was employed in Reference 4. This involved simply making the substitution

$$x \to x - x_{min}$$

in the distribution function. If a cutoff at larger sizes is desirable, one needs another means of making the logarithmic term infinite. This can be easily supplied by first writing

$$\ln x = - \ln \frac{1}{x}$$

and making the substitution

$$\frac{1}{x} \to \frac{1}{x} - \frac{1}{x_{max}}$$

which is equivalent to making the substitution

$$x \to \frac{x}{1 - x/x_{max}}$$

An obvious way to incorporate both cutoffs simultaneously is to make the single substitution

$$x \to \frac{x - x_{min}}{1 - x/x_{max}} .$$

Appendix A describes how truncated graphs can be constructed from the parameters μ_M and σ with a minimum of effort.

Comparison

Figure 2 compares the lognormal and power-law distributions in truncated form. For comparison, all distributions are truncated at 1 μ and 1000 μ. With the truncated power-law distribution it is possible to show the graph for $q = 4$. Therefore, Fig. 2 shows the full extent of uncertainty due to the uncertainty in q. For the case of $q = 4$:

$$P_M \{0,x\} = \int_1^x d \ln x \Big/ \int_1^{1000} d \ln x$$

$$= \frac{1}{3} \log_{10} x(\mu)$$

It is noteworthy that this $q = 4$ distribution is also the distribution of surface that would correspond to the $q = 3$ mass or volume distribution.

The truncated lognormal distributions are shown for the assumptions of both constant volume-specific activity and constant surface-specific activity. The former lies well within the range of power-law distributions.

Variation in x_{max}

As indicated above, the value of x_{max} is not well known. The above equations are easily adapted to other values. Choosing 500 and 2000 for illustrations only gives the values shown in Table 1. The sensitivity to x_{max} is seen to be strongly dependent on the value of q. Thus, the effect on $P_M \{0,x\}$ ranges from a factor of 1.2 for $q = 4$ up to a factor of four for $q = 3$.

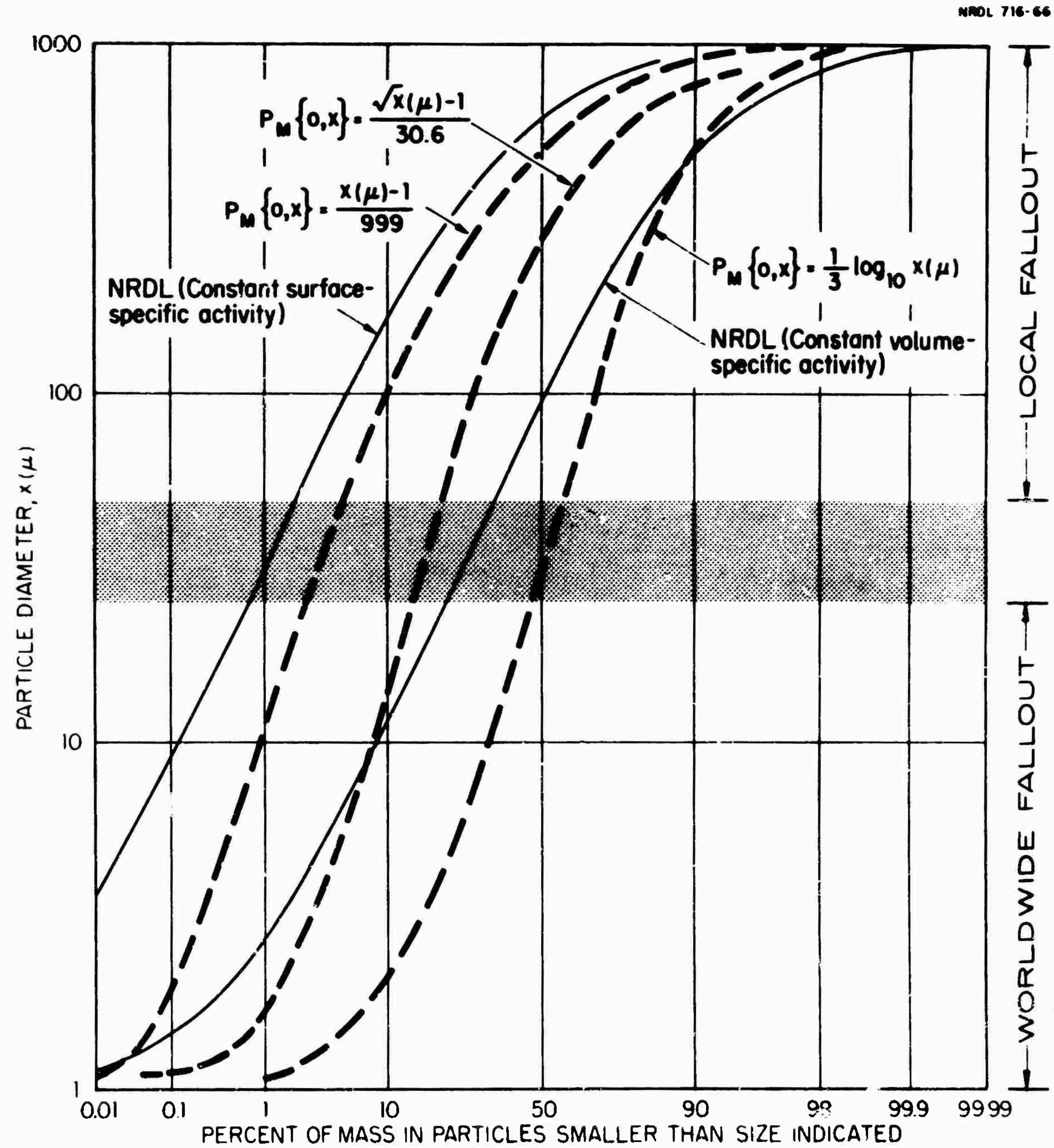

Fig. 2 Comparison of Truncated Distributions; Effect of Uncertainty in Exponent

TABLE 1

Effect of Varying x_{max} on Power-Law Equations with Various Exponents

Cutoff Diameter (x_{max})	$P_M\{0,x\}$		
	q = 3.0	q = 3.5	q = 4.0
500 µ	$\frac{x-1}{499}$	$\frac{\sqrt{x-1}}{\text{[illegible]}1.4}$	$\frac{1}{2.7}\log_{10} x$
1000 µ	[illegible]	$\frac{\sqrt{x-1}}{30.6}$	$\frac{1}{3}\log_{10} x$
2000 µ	$\frac{x-1}{1999}$	$\frac{\sqrt{x-1}}{43.7}$	$\frac{1}{3.3}\log_{10} x$

SIGNIFICANCE

As Fig. 1 illustrates, for the particle size range of 10-800 µ, the extended power-law distributions for exponents of 3.0 and 3.5 lie between the lognormal curves based upon constant surface-specific activity and constant volume-specific activity. Below 10 µ, the divergence of the power-law distribution from the area bounded by the lognormal lines is at most about 4 % of the total mass. Above 800 µ, this divergence varies with large particle cutoff, and for a cutoff of 2000 µ the divergence is less than 5 %.

Figure 2 shows that, for truncated distributions, the lognormal curve based on the assumption of constant volume-specific activity lies well within the range of uncertainty in power-law exponents and does not differ greatly from the curve based on an exponent of 3.5. Although the lognormal curve for the assumption of constant surface-specific activity lies outside the range of uncertainty in power-law exponents, it does not differ greatly from the curve based on an exponent of 3.0.

Figure 3 is a striking illustration of the similarity that can be achieved between the truncated power-law distribution and the truncated lognormal distribution. This figure compares the q = 3.5 power-law

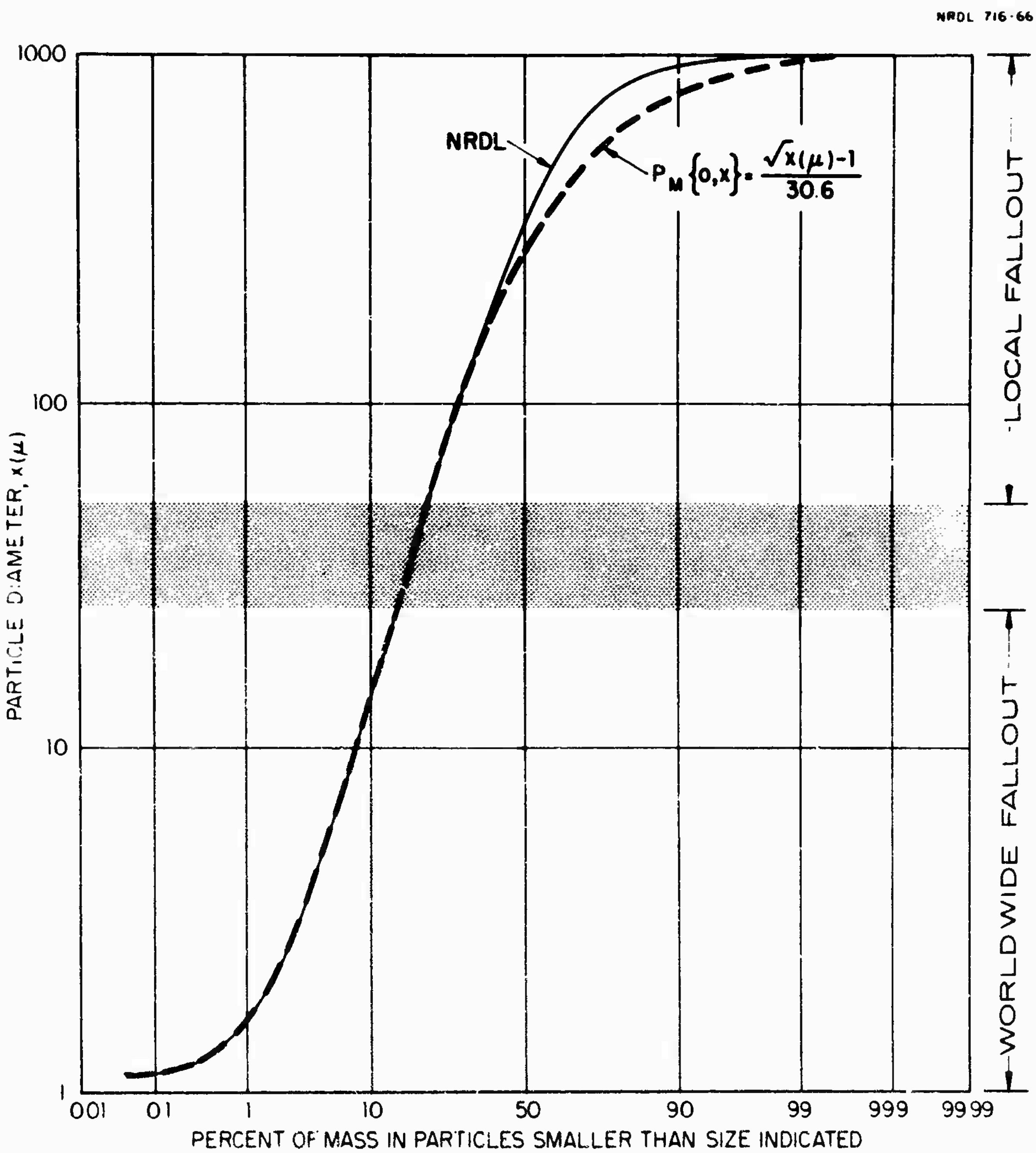

Fig. 3 Achievable Similarity Between Power-Law and NRDL Lognormal Distributions. Lognormal parameters are $\underline{x}_M = 500\ \mu$, $x_{M^-} = 30\ \mu$, $x_{min} = 1\ \mu$, $x_{max} = 1000\ \mu$.

distribution with the $\underline{x}_M = 500\ \mu$, $x_{M^-} = 30\ \mu$ lognormal distribution. The agreement is such that if one curve is correct, the other will never be proved wrong. Presumably, equal similarity could be achieved with curves for other values for q. Presumably also, the agreement in Fig. 3 could be made still closer by either (a) a different choice of σ and μ_M or (b) an equally reasonable method of cutting off the lognormal distribution at 1000 μ, or (c) cutting off the lognormal distribution at some equally defensible value like 950 μ.

Thus it appears that the differences between the two approaches are trivial. The lognormal distribution has the esthetic advantage of an observationally confirmed theoretical basis in the case of airburst debris. If truncation is required, the power-law distribution has the practical advantage of simplifying further calculations of particle surface distribution. Planned sensitivity analyses need not incorporate power-law distributions explicitly, but may include them as a matter of convenience.

REFERENCES

1. I. J. Russell, "Particle Size Distribution (U)," in A Critique of Land-Surface Fallout Models (U), Air Force Weapons Laboratory, AFWL-TR-65-76, February 1966 (Classified).

2. E. C. Freiling, "Fractionation in Surface Bursts," in Radioactive Fallout from Nuclear Weapons Tests, U. S. Atomic Energy Commission, Division of Technical Information, TID-7632, February 1962.

3. P. D. LaRiviere, S. L. Brown, J. D. Sartor and C. F. Miller, Local Fallout from Nuclear Test Detonations (U). Vol. V. Transport and Distribution of Local (Early) Fallout from Nuclear Weapons (U), Stanford Research Institute, SRI-4-3338 (DASA 1251, NDL-TR-34), 1965 (Classified).

4. E. C. Freiling, "Fractionation. III. Estimation of Degree of Fractionation and Radionuclide Partition for Nuclear Debris," U. S. Naval Radiological Defense Laboratory, USNRDL-TR-680, 12 September 1960.

APPENDIX A

GRAPHICAL CONSTRUCTION OF TRUNCATED LOGNORMAL DISTRIBUTIONS

Consider first the region of sizes less than $\underline{x}$ and assume that the effect of x_{max} is negligible in this region. The truncated curve can be calculated from the extended curve by simply replotting percentages. Thus, for a cutoff of 1 μ, the value of the extended curve for 1 μ is replotted at 2 μ, the value for the extended curve at 2 μ is replotted at 3 μ, etc.

In the size region above $\underline{x}$, assuming the effect of x_{min} can be neglected, the procedure is similar. Thus, for a value of 1000 μ for x_{max}, if one wishes to plot a value for 900 μ, one simply plots the value corresponding to 900/(1 - 900/1000) or 9000 μ on the extended curve.

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